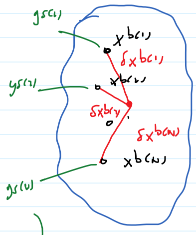
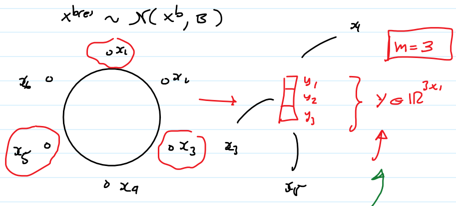
# Data Assimilation

miércoles, 2 de septiembre de 2020 01:42 p. m.




$\Delta x \in \mathbb{R}^{n\times N}$ $\quad \delta x^{b(e)} \sim \mathcal{N}(0, B)$

$x^{b(e)} \sim \mathcal{N}(x^b, B)$




$\boxed{m=3}$

$y \in \mathbb{R}^{3\times 1}$

$y^{s(e)} \sim \mathcal{N}(y, R)$

DATA ERROR COVARIANCE MATRIX

## BENCHMARK

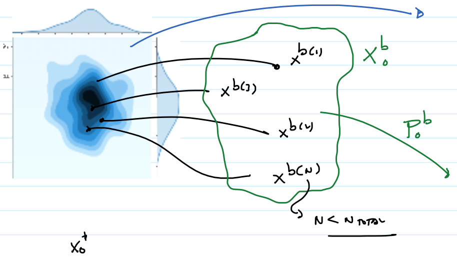
$$X^*_{RANDOM} \xrightarrow{MODEL} X^*_{-2} \xrightarrow{MODEL} X^*_{-1} \xrightarrow{MODEL} X^*_0$$

$$X^b_{RANDOM,\,-2} \xrightarrow{MODEL} X^b_{-1} \xrightarrow{MODEL} X^b_0 \quad \text{FREE RUN}$$

$$\hat{X}^b_{RANDOM,\,-1} \xrightarrow{MODEL} \hat{X}^b_0$$

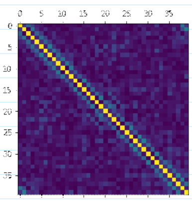
$\hat{X}^b_0 \in \mathbb{R}^{n\times N_{TOTAL}}$ $\quad N_{TOTAL} = 200$

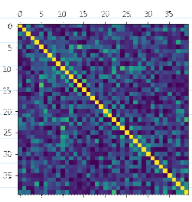
$\hat{P}^b_0$




$N < N_{TOTAL}$

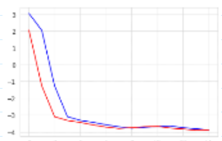
$X^*_0$





(+) FOR $N = N_{TOTAL}$ ,

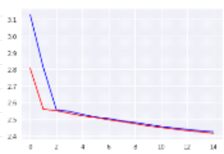
$\|x^b_k - x^*_k\| \approx 10^{-4}$



$N = 200$

(+) FOR $N = 20$

$\|x^b_k - x^*_k\| \approx 10^{2.4}$



$N = 20$

(+) SAMPLING CAN IMPACT THE QUALITY OF ANALYSIS CORRECTION (+) OJO

$R \in \mathbb{R}^{m\times m}$ $\quad m \sim \mathcal{O}(10^6)$

(+) $$x^a = x^b + P^b H^T (R + H P^b H^T)^{-1} D \qquad D = y^s - H x^b$$

$P^b \in \mathbb{R}^{n\times n}$ $\quad n \sim \mathcal{O}(10^8)$

LONG FLOAT COMPUTATION (+) (/)

(+) $$x^a = x^b + P^b H^T Z \qquad (R + H P^b H^T) Z = D$$

LONG FLOAT COMPUTATION ① ②

$(+)\quad x^a = x^b + P^b H^T Z \qquad (R + H P^b H^T) Z = D$

$\underbrace{(R + H P^b H^T)}$ $\boxed{\mathcal{O}(m^3)}$

$n \times n \qquad n \times m \qquad m \times N$

$\mathcal{O}(n^2 N + nmN) \qquad \mathcal{O}(nmN)$

$(+)\quad n \sim \mathcal{O}(10^8)$

$n \times n \qquad n \times N$

$\mathcal{O}(n^2 N)$

DON'T BUY THAT SH

$(+)\quad P^b = \frac{1}{N-1} \Delta X \Delta X^T \qquad \Delta X \in \mathbb{R}^{n \times N}$

$n \times n \qquad n \times N \qquad N \times n$

$N << n$

$\text{s/}\ \Delta X = U \Sigma V^T$

$\Delta X = \sum_{e=1}^{N-1} \sigma_e \cdot u_e \cdot v_e^T$

RANK-DEFICIENT

$P^b = \frac{1}{N-1} U \Sigma^2 U^T \qquad \begin{cases} \sigma_e > 0 & e < N \\ \sigma_e = 0 & N \le e \le n \end{cases}$

$\Delta X = U\,\Sigma\,V^T \ (n \times N,\ N \times N,\ N \times N) \longrightarrow P^b = \underbrace{U \quad \Sigma^2 \quad U^T}_{} \Big/ N-1 \ (n \times N,\ N \times N,\ N \times n)$

$(+)\quad x^a = x^b + P^b H^T Z \qquad (R + H P^b H^T) Z = D$

$H P^b H^T = Q Q^T \qquad Q = H \cdot \overbrace{\Delta X \frac{1}{\sqrt{N-1}}}^{\hat{\Delta X}} \qquad \checkmark \in \mathbb{R}^{m \times N}$

$Q Q^T = \left[ H \Delta X / \sqrt{N-1} \right] \left[ H \Delta X / \sqrt{N-1} \right]^T$

$\hat{\Delta X} = \frac{1}{\sqrt{N-1}} \Delta X \in \mathbb{R}^{n \times N} \qquad = H \left( \frac{1}{N-1} \Delta X \Delta X^T \right) H^T = H P^b H^T$

$x^a = x^b + \hat{\Delta X}\, \hat{\Delta X}^T H^T Z$

$= x^b + \hat{\Delta X} (\underbrace{H \hat{\Delta X}}_{Q})^T Z$

$= x^b + \hat{\Delta X}\, Q^T Z \qquad (R + \overbrace{H P^b H^T}^{Q Q^T}) Z = D$

$n \times N \qquad N \times m \qquad m \times N$

$(R + Q Q^T) Z = D$

```
DX_k = XB_k-np.outer(xb_k,np.ones(N));
DXG_k = 1/(np.sqrt(N-1))*DX_k;
#Innovation matrix
Q_k = H_k@DXG_k;
IN_k = R_k + Q_k@Q_k.transpose();
Z_k = np.linalg.solve(IN_k,D_k);
```

$\mathcal{O}(N^2 \cdot m)$

$(+)\quad N \times N$

$\mathcal{O}(n \cdot N^2)$

$\mathcal{O}(N^2 \cdot m + N^2 \cdot n)$

m AND n ARE LINEAR

① $x^a = x^b + P^b H^T Z \rightarrow \mathcal{O}(N^2 (n^2 + nm))$

② $x^a = x^b + \hat{\Delta X} Q^T Z \rightarrow \mathcal{O}(N^2 (m + n))$ RANK-DEFICIENT

$\hat{\Delta X} = \frac{1}{\sqrt{N-1}} \Delta X \qquad Q = H \hat{\Delta X} \in \mathbb{R}^{m \times N}$

$\in \mathbb{R}^{n \times N}$

$$\hat{\Delta x} = \frac{1}{\sqrt{N-1}} \Delta X \in \mathbb{R}^{n \times N} \qquad Q = H \hat{\Delta X} \in \mathbb{R}^{m \times N}$$

$$X^a - X^b = \hat{\Delta X}\, \underbrace{Q^T Z}_{\Phi \,\in\, \mathbb{R}^{N \times N}}$$

$$X^a - X^b = \hat{\Delta x}\, \Phi \Rightarrow x^{a(e)} - x^{b(e)} = \hat{\Delta x} \cdot \phi^{(e)}$$

RANK

$$\delta x^{a(e)} = \hat{\Delta x}\, \phi^{(e)}$$

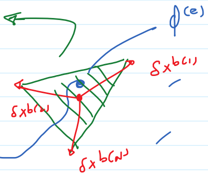
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\delta x^{a(e)} \in \text{SPAN}\left\{ \frac{1}{\sqrt{N-1}} \delta x^{b(1)}, \frac{1}{\sqrt{N-1}} \delta x^{b(2)}, \ldots, \frac{1}{\sqrt{N-1}} \delta x^{b(N)} \right\}$$

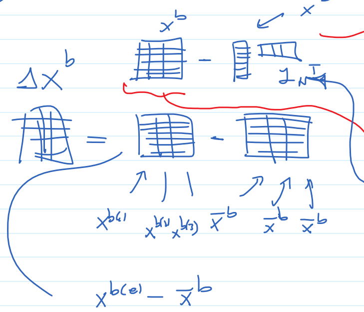
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\in \text{RANGE}\left( \hat{\Delta X} \right) \quad \frac{1}{\sqrt{N-1}} \Delta X$$

$$\delta x^{a(e)} \in \text{RANGE}\left( \Delta X \right)$$

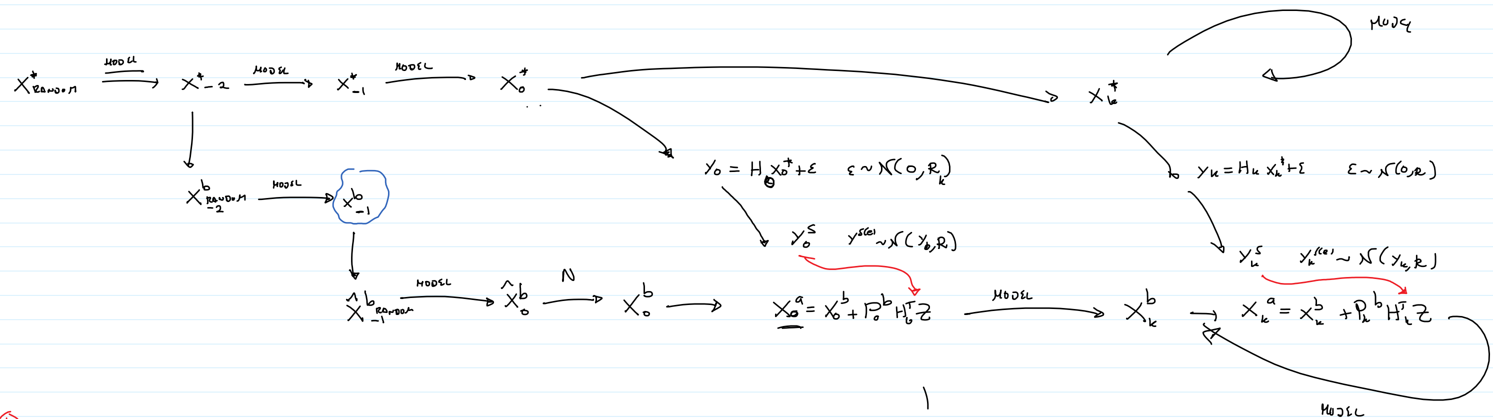



(+) $\Delta X^b = X^b - \bar{x}^b \cdot 1_N^T$




```
DX_k = XB_k-np.outer(xb_k,np.ones(N));
```

## Benchmark




(+) LINEAR SYSTEM OF EQUATIONS

$$\left( R + H P^b H^T \right) Z = D$$

$$\left( R + Q Q^T \right) Z = D$$

$$\mathbb{E}(\varepsilon_i\, \varepsilon_i^T) = 0$$

$$R + H P^b H^T = \left( R^{1/2} + H \hat{\Delta x}^b \right)\left( R^{1/2} + H \hat{\Delta x}^b \right)^T$$

$\eta \sim \mathcal{N}(0, I)$

$\ldots \sim \mathcal{N}(0, RR^T)$

$\mathbb{E}(\varepsilon_2 \varepsilon_1^T) = 0$

$R + HP^bH^T = (R^{1/2} + H\hat{\Delta x}^b)(R^{1/2} + H\hat{\Delta x}^b)^T$

$\boxed{\eta \sim \mathcal{N}(0, I) \quad B\eta \sim \mathcal{N}(0, BB^T)}$

$\mathbb{E}(\varepsilon_1 \cdot \varepsilon_2^T) = 0 \quad \operatorname{cov}(\varepsilon_1, \varepsilon_2) = \underset{N}{0} \in \mathbb{R}^{m\times N}$

$\varepsilon_1 \sim \mathcal{N}(0, I_m) \qquad \varepsilon_2 \sim \mathcal{N}(0, I_N)$

$\underbrace{R^{1/2}\varepsilon_1}\sim \mathcal{N}(0, R) \qquad H\hat{\Delta x}^b\varepsilon_2 \sim \mathcal{N}(0, H\underbrace{\hat{\Delta x}^b \hat{\Delta x}^{bT}}_{P^b}H^T)$

$\underbrace{H\hat{\Delta x}^b\varepsilon_2} \sim \mathcal{N}(0, HP^bH^T)$

$\mathbb{E}\left([R^{1/2}\varepsilon_1 + H\hat{\Delta x}^b\varepsilon_2][R^{1/2}\varepsilon_1 + H\hat{\Delta x}^b\varepsilon_2]^T\right)$ (+) MANDEL

$= \mathbb{E}\left(R^{1/2}\varepsilon_1\varepsilon_1^T R^{1/2} + R^{1/2}\varepsilon_1\varepsilon_2^T\hat{\Delta x}^{bT}H^T + H\hat{\Delta x}^b\varepsilon_2\varepsilon_1^T R^{T/2} + H\hat{\Delta x}^b\varepsilon_2\varepsilon_2^T\hat{\Delta x}^{bT}H^T\right)$

$= R^{1/2}\underbrace{\mathbb{E}(\varepsilon_1\varepsilon_1^T)}_{I_m}R^{1/2} + \underbrace{R^{1/2}\underbrace{\mathbb{E}(\varepsilon_1\varepsilon_2^T)}_{0\in\mathbb{R}^{m\times N}}\hat{\Delta x}^{bT}H^T}_{0} + \overbrace{H\hat{\Delta x}^b\underbrace{\mathbb{E}(\varepsilon_2\varepsilon_1^T)}_{0\in\mathbb{R}^{N\times m}}R^{1/2}}^{0}$

$+ H\hat{\Delta x}^b\underbrace{\mathbb{E}(\varepsilon_2\cdot\varepsilon_2^T)}_{I_N}\hat{\Delta x}^{bT}H^T$

$= R + H\underbrace{\hat{\Delta x}^b\hat{\Delta x}^{bT}}_{P^b}H^T = R + HP^bH^T$ ✓ OK

(+) $\underbrace{R} + \underbrace{H\hat{\Delta x}} = U\Sigma V^T \qquad {}_m\overset{m}{\square} + {}_m\overset{N}{\square} = U\Sigma V^T$

$R + HP^bH^T = U\Sigma V^T V\Sigma U^T = U\Sigma^2U^T$

DIAGONAL

$(R + HP^bH^T)Z = D \Rightarrow [\underbrace{U}\Sigma^2\underbrace{U^T}]Z = D$

ORTHONORMALS

$\boxed{Z = U\Sigma^{-2}U^TD} \qquad \boxed{UU^T = U^TU = I}$

(+) SVD

${}_n\overset{N}{\square}$ $n >> N$ $\quad O(N^2 n)$ → LINEAL # VAR $\Rightarrow$ QR FACTORIZATION

${}_m\overset{m}{\square}$ $\quad O(m^3)$ ?

$\boxed{{}_m\overset{m}{\boxed{R}} \quad {}_m\overset{N}{\boxed{R}} + {}_n\overset{N}{\boxed{H\Delta x}} = U\Sigma V^T}$ STAT

(+) SVD $\quad (R + QQ^T)Z = D$

$Q \in \mathbb{R}^{m\times N} \qquad Q = H\hat{\Delta x}^b \qquad \hat{\Delta x}^b = \frac{1}{\sqrt{\ldots}}\Delta x^b$

$$Q \in \mathbb{R}^{m \times N} \qquad Q = H\,\hat{\Delta X}^b \qquad \hat{\Delta X}^b = \frac{1}{\sqrt{N-1}}\Delta X^b$$

$$Q = U \Sigma V^T \Rightarrow \Theta(m \cdot N^2) \qquad \Rightarrow \Delta X^b = X^b - \bar{x}^b 1_N^T$$

LINEAL
IN REGARD TO # OBS

$R =$ 

(I)
$$R + QQ^T = R + (U\Sigma V^T)(U\Sigma V^T)^T$$
$$= R + U\Sigma^2 U^T \rightarrow \Theta(mN^2)$$

(1) R IS DIAGONAL
(2) R IS BLOCK-DIAGONAL
(3) R IS EASY TO DECOMPOSE

$\mathcal{N}(0, \sigma^2)$

$\hookrightarrow R = \sigma^2 I$

$$R + QQ^T = \sigma^2 I + U\Sigma^2 U^T = U[\sigma^2 \cdot U^T + \Sigma^2 U^T]$$

$$= \underbrace{U}_{\text{ORTHONORMAL}}[\underbrace{\sigma^2 I}_{\text{DIAGONAL}} + \underbrace{\Sigma^2}_{\text{DIAGONAL}}]U^T \quad \text{ORTHONORMAL}$$

$\hat{\Sigma}^2 = \sigma^2 I + \Sigma^2$

$$= \underset{m}{\boxed{U}}\ \overset{N}{\underset{N}{\boxed{\hat{\Sigma}^2}}}\ \overset{m}{\underset{N}{\boxed{U^T}}} = \underline{\underline{R + HP^bH^T}}$$

$$[R + HP^bH^T]Z = D \Rightarrow U\hat{\Sigma}^2 U^T Z = D$$

$$\boxed{Z = U\hat{\Sigma}^{-2}U^T D}$$

(I)
$$X^a = X^b + \hat{\Delta X}^b Q^T Z$$
$$= X^b + \hat{\Delta X}^b [U\Sigma V^T]^T [U\hat{\Sigma}^{-2}U^T D]$$
$$= X^b + \hat{\Delta X}^b V \Sigma \underbrace{U^T U}_{I} \hat{\Sigma}^{-2} U^T D$$
$$=$$

$$\hat{\Sigma}^{-2} \rightarrow (\sigma^2 I + \Sigma^2)^{-2}$$

$$= X^b + \hat{\Delta X}^b V \Sigma \hat{\Sigma}^{-2} U^T D$$

$$= X^b + \hat{\Delta X}^b V \cdot \mathrm{diag}\left(\frac{\sigma_i}{\sigma_{OBS} + \sigma_i^2}\right) U^T D$$

$\Sigma \qquad \hat{\Sigma}^{-2}\ (-1)$

$\sigma_1\ \sigma_2\ \ldots\ \sigma_N$

$(\sigma + \sigma_1^2)^{-1}$
$(\sigma + \sigma_2^2)^{-1}$
$(\sigma + \sigma_N^2)^{-1}$

$\Rightarrow$

$\sigma_1 / \sigma_1^2 + \sigma$
$\sigma_2 / \sigma_2^2 + \sigma$
$\sigma_N / \sigma_N^2 + \sigma$